Higher Order Winkler Analytical Solutions for Flexible Piles

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ABSTRACT : Novel analytical solutions of the Winkler type are derived for the response to lateral dynamic loads of a flexible elastic pile embedded in inhomogeneous soil, based on three soil constants instead of one in the classical formulation. This extension allows for a more rational calibration of the model against rigorous solutions such as finite elements or boundary elements, by matching all three stiffness constants (for swaying, rocking and cross-swaying rocking) at the pile head. This approach leads to a more realistic representation of pile-soil interaction and a better estimation of the internal forces along the pile – notably peak pile bending moments. Both inertial and kinematic interaction are examined, due to harmonic head loads and vertically propagating shear waves. Closed form solutions are obtained for the: (1) stiffness coefficients, (2) kinematic response coefficients, (3) peak shear forces and bending moments. The method does not lead to a significant increase in the complexity of the analysis, as the order of the governing differential equation and the boundary conditions at the two ends of the pile are the same as in the classical problem. A novel geometric interpretation of the three elastic constants is provided.

1. INTRODUCTION

New analytical solutions based on the Winkler model of soil reaction are presented for flexible elastic piles embedded in inhomogeneous elastic soil. Contrary to the classical solutions which utilize a single soil constant (coefficient of subgrade reaction), the proposed method employs three soil constants which generate shear tractions, external moments and internal moments on the pile, proportional to displacement, rotation and curvature, respectively. The use of these independent constants facilitates the calibration of the model against rigorous numerical solutions based on continuoum representation of the soil, by matching all three stiffness coefficents (against swaying, rocking and cross-swaying rocking) at the pile head. In addition to providing a more rational representation of soil-pile interaction, the model leads to a better estimation of bending moments along the pile. Also, the model does not increase the complexity of the analysis, since the order of the differential equation and the boundary conditions do not change – contrary to corresnponding gradient theories of Continuum Mechanics. It is proven by dimensional arguments that the soil coefficients are dependent on soil-pile stiffness contrast, Poisson's ratio and boundary conditions at pile head. Closed-form solutions for these three soil constants and comparisons with finite element solutions are presented. This method is a generalization of the models of Hetenyi (1946) and Pasternak (1954), which employ only two soil constants.

The Winkler's model (1867) – which was first developed for the idealization of the soil in soil-structure interaction problems – is based on the reasonable but approximate modeling of the soil as a system of similar, independent, linear springs which are represented by only one coefficient of sub-grade reaction, k.

According to the abovementioned theory, the static stiffness at the head of a flexible pile in homogeneous soil, is computed by the simple equations (Hetenyi, 1946):

$$K_{HH} = 4E_p I\,\lambda^3, \quad K_{HR} = 2E_p I\,\lambda^2, \quad K_{RR} = 2E_p I\,\lambda \qquad \text{(1a,b,c)}$$

$$\lambda = \left[k/\left(4E_p I\right)\right]^{1/4} \qquad \text{(2)}$$

where λ is the Winkler coefficient (dimensions 1/Length) and E_pI the flexural stiffness of the pile (dimensions Force x Length^2). The expressions in Equations 1a, b and c give the coefficients in swaying, rocking of the pile head and cross-swaying rocking, respectively.

For a free-head pile, the lateral stiffness for a force applied on top under zero bending is given by

$$\mathrm{K_H} = 2E_p I\,\lambda^3 \qquad \text{(3)}$$

which is derived by combination of the above equations and predicts that the stiffness for a free-head pile is equal to 50% of the stiffness for a fixed-head pile (in contrast to 25% for a column).

It is known that the idealization of springs with only one constant *k*, compared with the analysis of the soil as a continuum media, gives inconsistent results for soil-pile interaction. This is due to the Winkler's model incapability to capture the cross-correlation between the springs. So, Wieghardt (1922) followed by Filonenko-Borodich (1940), Hetenyi (1946), Pasternak (1954) and Vlasov-Leontiev (1966) proposed improvements to the original model with the introduction of a second coefficient (k_φ), which can be interpreted either through a membrane which connects the base of Winkler springs (Hetenyi 1946) or through rocking springs distributed along the pile (Shanchez-Salinero 1982).

2. PROBLEM DEFINITION

The problem to be analyzed is presented in Figure 1. Pile embedded in homogeneous soil, subjected to lateral load and moment at its head. The pile is simulated as a linear elastic, homogeneous, cylindrical, Euler-Bernoulli beam with constant diameter *d*, length *L* and elastic modulus E_p. Moreover, the pile is considered to be flexible, so that it will not deflect for its entire length, but only up to its «active» length, L_α (Randolph, 1981) beyond which it does not respond to the lateral force at its head, thus its real length does not affect its stiffness. The soil is assumed to be linear, viscoelastic with elastic modulus E_s and Poisson's ratio ν_s. The contact at the interface of pile-soil is considered to be perfect, without sliding or separation between these two materials.

For static loading conditions, the main dimensional parameters of the problem are the length of the pile, *L*, its diameter, *d*, its elastic modulus, E_p, and the elastic modulus of the soil, E_s. The fundamental dimensions are length [L] and force [F]. Consequently, the number of main dimensional parameters is *M*=4 and the equivalent number of fundamental dimensions is *N*=2. Applying Buckingham's theorem (1914), *M*–*N*=2 dimensionless parameters are needed for the description of our problem. These parameters are the ratio of the length of the pile over its diameter, L/d and the stiffness of the pile-soil, E_p/E_s. Moreover, the dimensionless ratio of the Poisson ratio of the pile, ν_p, and the soil, ν_s are included in this solution.

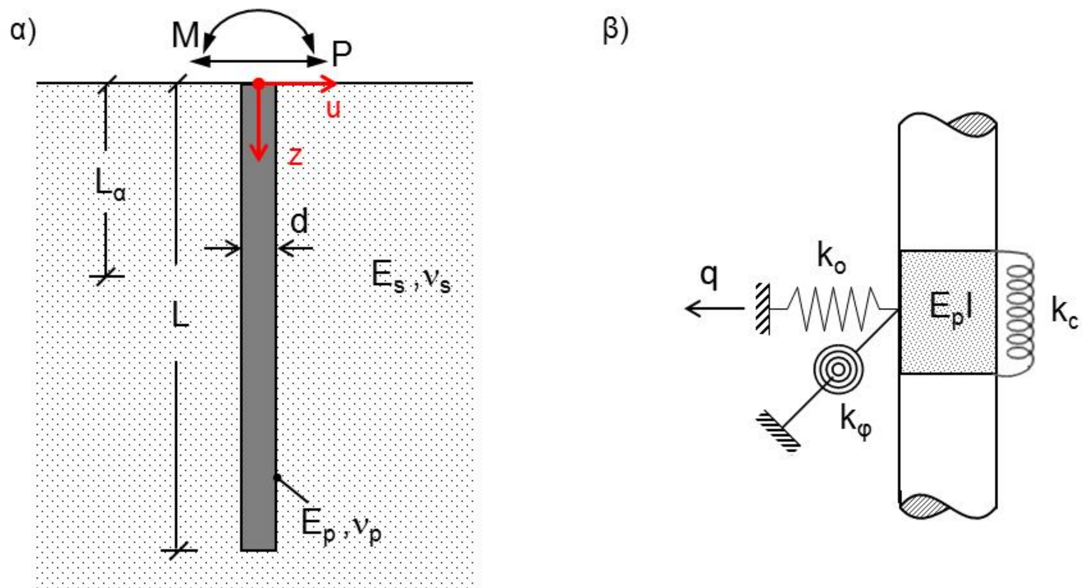

Figure 1. (a) Problem definition, (b) Infinitesimal pile segment of three-parameter Winkler model

3. FORWARD ANALYSIS

For the description of the behavior of the pile, the abovementioned Winkler's model is proposed, which includes three coefficients of sub-grade reaction k_o, $k_φ$ και k_c (Figure 1b). In the context of this theory, the equation of equilibrium of the pile is

$$\left(E_p I - k_c\right)u^{(4)} - k_\varphi u^{(2)} + k_o u = 0 \tag{4}$$

where the superscript $^{(\,)}$ represents differentiation to the variable (z). The three soil constants create shear forces and moments in the pile segment, which are proportional to the range of curvature, rotation and deflection of the pile, respectively. The constants of Winkler springs k_o, $k_φ$ και k_c are related to the soil stiffness through the equations given below, which satisly the dimensions of each term in Equation 4.

$$\mathrm{k}_o = \delta_o\, E_\mathrm{s}, \quad \mathrm{k}_\phi = \delta_\phi\, E_\mathrm{s}\, \mathrm{d}^2, \quad \mathrm{k_c} = \delta_\mathrm{c}\, E_\mathrm{s}\, \mathrm{d}^4 \tag{5a,b,c}$$

where $δ_o$, $δ_φ$ and $δ_c$ are dimensionless Winkler constants, the values of which are examined below. It should be noted that Equation 4 can be derived based on the alternative hypothesis that soil reaction depends on the higher-order derivatives of deflection

$$q = k_o\, u - k_\phi\, u^{(2)} - k_c\, u^{(4)} \tag{6}$$

an assumption which does not include distributed moments and a solution which does not include derivatives of odd order for obvious reasons. It should be highlighted that although the equilibrium equations are the same, the above assumption is not equivalent to the higher order Winkler model as shown in Figure 1b (Agapaki 2014).

According to the three-parameter Winkler model, the stiffness of a pile of infinite length embedded in homogeneous soil is computed in the following equations

$$K_{HH} = 2(E_p I)'\lambda\left(\lambda^2 + \mu^2\right) \tag{7}$$

$$K_{HR} = -(E_p I)'\left(\lambda^2 + \mu^2\right) \tag{8}$$

$$K_{RR} = 2(E_p I)'\lambda \tag{9}$$

where

$$\left(E_p I\right)' = E_p I - \delta_c E_s d^4 \tag{10}$$

and λ, μ the Winkler parameters (with dimensions 1/Length) and E_pI the flexural stiffness of the pile. These two Winkler parameters are obtained by the relations below (Shanchez-Salinero 1982, Agapaki 2014):

$$\lambda^4 = \frac{k_o}{4\left(E_p I\right)'}\left(1 + \frac{k_\varphi}{2\sqrt{(E_p I)' k_o}}\right)^2 \tag{11}$$

$$\mu^4 = \frac{k_o}{4\left(E_p I\right)'}\left(1 - \frac{k_\varphi}{2\sqrt{(E_p I)' k_o}}\right)^2 \tag{12}$$

For a free-head pile, the lateral stiffness of a pile subjected to a lateral force with no bending moment at its head is given by

$$K_H = \frac{\left(E_p I\right)'\left(3\lambda^4 + 2\lambda^2\mu^2 - \mu^4\right)}{2\lambda} \tag{13}$$

Lastly, for $\lambda=\mu$ and $(E_pI)'=E_pI$, the above relations are similar to Equations 1 and 3.

4. BACKWARD ANALYSIS

Considering the stiffnesses of the pile K_{HH}, K_{HR} and K_{RR}, as known, it is possible to determine the constants δ_o, δ_φ και δ_c of the model following backward analysis. More specifically, the following flexural stiffness of the pile is derived from Equations 7- 9

$$\left(E_p I\right)' = \frac{K_{RR}}{2\lambda} = \frac{K_{RR}\left(-K_{HR}\right)}{K_{HH}} \tag{14}$$

which is valid for the typical Winkler model and allows the constant δ_c, through Equation 10, to be based on stiffness coefficients at the pile head to be determined as

$$\delta_c = \frac{E_p I}{E_s d^4}\left(1 - \frac{K_{\mathrm{RR}}\,|K_{HR}|}{E_p I\, K_{\mathrm{HH}}}\right) \tag{15}$$

The use of absolute value for $|K_{HR}|$ in Equation 15 is used for clockwise and counterclockwise reference systems. Adding Equations 11 and 12 by parts and after combining Equations 7 and 9, the constant k_o is given

$$\mathrm{k}_o = \left(E_p I - \delta_{\mathrm{c}}\, E_s\, \mathrm{d}^4\right)\left(\frac{K_{\mathrm{HH}}}{K_{\mathrm{RR}}}\right)^2 \tag{16}$$

Combining Equations 5a and 16, we get

$$\delta_{\mathrm{o}} = \left(\frac{E_p I}{E_s \mathrm{d}^4} - \delta_{\mathrm{c}}\right)\left(\frac{K_{\mathrm{HH}}\mathrm{d}^2}{K_{\mathrm{RR}}}\right)^2 \tag{17}$$

Dividing the parts of Equations 11 and 12, the constant k_φ is derived as

$$\mathrm{k}_{\varphi} = 2(E_p I)'(\lambda^2 - \mu^2) \tag{18}$$

So, taking into account the Equations 7-10 and 14, the constant δ_φ is given from Equation 18 as a function of stiffness coefficients

$$\delta_{\varphi} = 2\left(\frac{E_p I}{E_s \mathrm{d}^4} - \delta_{\mathrm{c}}\right)\left[\left(\frac{E_p I}{E_s \mathrm{d}^4} - \delta_{\mathrm{c}}\right)^{-2}\left(\frac{K_{\mathrm{RR}}}{2E_s \mathrm{d}^3}\right)^2 - \left(\frac{K_{\mathrm{HH}}\mathrm{d}^2}{K_{\mathrm{RR}}}\right) + \left(\frac{K_{\mathrm{HH}}\mathrm{d}}{2\,K_{\mathrm{HR}}}\right)^2\right] \tag{19}$$

Given that the stiffness coefficients are written in the more useful form

$$K_{HH} = \chi_{HH} E_s\, \mathrm{d}, \quad K_{HR} = \chi_{HR} E_s\, \mathrm{d}^2, \quad K_{RR} = \chi_{RR} E_s\, \mathrm{d}^3 \tag{20a,b,c}$$

from Equations 15, 17, 19 and 20 we get the following simpler expressions

$$\delta_{\mathrm{o}} = \frac{\chi_{HR}\,\chi_{HH}}{\chi_{RR}} \tag{21}$$

$$\delta_{\varphi} = 2\left(\frac{\chi_{RR}\,\chi_{HR}}{\chi_{HH}}\right)\left[\left(\frac{\chi_{HH}}{\chi_{HR}}\right)^2 \frac{1}{2} - \left(\frac{\chi_{HH}}{\chi_{RR}}\right)\right] \tag{22}$$

$$\delta_{\mathrm{c}} = \frac{E_p I}{E_s \mathrm{d}^4}\left(1 - \frac{E_s \mathrm{d}^4 \chi_{RR}\,\chi_{HR}}{E_p I\,\chi_{HH}}\right) \tag{23}$$

where χ_{HH}, χ_{HR}, χ_{RR} are dimensionless constants, which are available in literature. More specifically, substituting in the above expressions the available relations for χ_{HH}, χ_{HR}, χ_{RR} of Syngros (2004) which have been modified as following

$$\chi_{HH} = 0.75\left(\frac{E_p}{E_s}\right)^{1/4}, \chi_{HR} = 0.21\left(\frac{E_p}{E_s}\right)^{1/2}, \chi_{RR} = 0.15\left(\frac{E_p}{E_s}\right)^{3/4} \quad \text{(24a,b,c)}$$

and moment of inertia for a cylindrical intersection of the pile $I_p = \pi d^4/64$, Equations 21-24 are simplified as following

$$\delta_o = 1, \ \ \delta_\varphi = 0.12\left(\frac{E_p}{E_s}\right)^{1/2}, \delta_c = 0.007\left(\frac{E_p}{E_s}\right) \quad (25)$$

which are sufficient for the application of the method. The equivalent expression for the constant δ_o of the one-parameter Winkler model is given as δ_o=1.17, which is reasonably greater than the equivalent coefficient in Equation 25 and is in good agreement with the proposed models of Syngros (2004) και Gazetas (1991).

5. MAXIMUM BENDING MOMENTS

For a free-head pile, the maximum bending moment is at a depth of $z=\pi/2\lambda$ and is derived by solving Equation 4 for boundary conditions $Q(0)=P$ and $M(0)=0$. It is proved that for the bending moment at the pile head we have (Agapaki, 2014)

$$M(z) = \frac{e^{-\mu z}\left(\mu^2 + \lambda^2\right)\sin(\lambda z)P}{\lambda\left(3\lambda^2 - \mu^2\right)} \quad (26)$$

The maximum bending moment for the three-parameter Winkler model is equal to

$$\frac{M_{max}}{Pd} = \chi_M\left(\frac{E_p}{E_s}\right)^{1/4} \quad (27)$$

where the dimensionless coefficient with χ_M being equal to 0.12. For the one-parameter Winkler model, the equivalent coefficient is equal to 0.13 indicating that the one-parameter model overestimates the bending moment at the pile head approximately about 10 % compared to the three-parameter one.

Regarding a fixed-head pile, Equation 27 is used with χ_M = 0.28 and the maximum bending moment is developed at the top. With relevant calculations for the one-parameter model, we get χ_M =0.32, which indicates that this model overestimates the bending moment at the pile head about approximately 20%.

6. RESULTS

The three Winkler constants are represented in Figure 2.

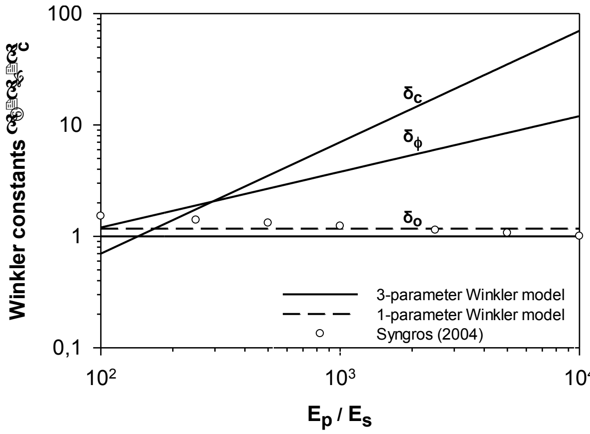

Figure 2. Winkler constants for a pile in homogeneous soil as function of E_p/E_s

We observe that the values of the dimensionless parameter δ_o in the three-parameter model are smaller than the one-parameter model, and do not change with the pile-soil stiffness, E_p/E_s. However, the parameters δ_φ and δ_c significantly increase with E_p/E_s.

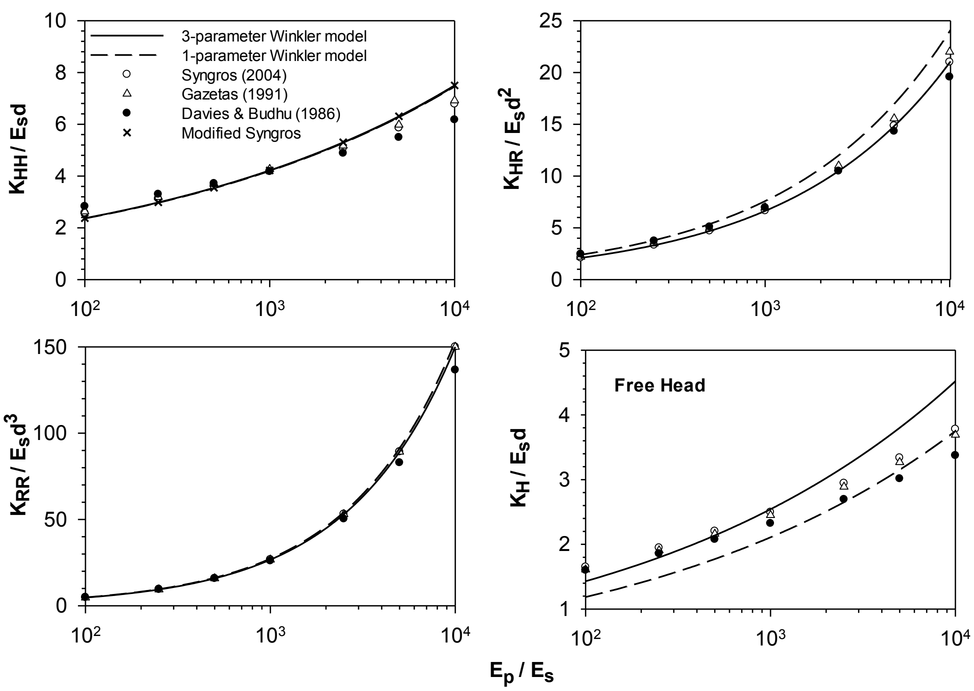

Figure 3. Pile stiffness coefficients for homogeneous soil (ν_p = 0.25, ν_s = 0.4)

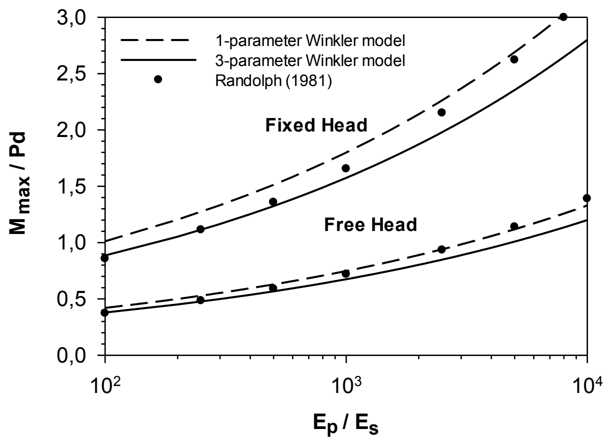

Figure 4. Maximum bending moments on a pile with different boundary conditions at the head under a horizontal load P, for different pile-soil stiffness contrasts (ν_s = 0.4)

In Figure 3 (a-c) the stiffness coefficients are presented for a fixed-head pile (Equations 1 & 7-9) as a function of the stiffness ratio E_p/E_s. It is observed that the relations for the stiffness coefficients of the three-parameter model are in very good agreement with the equivalent expressions in literature. Comparing with the one-parameter model, the stiffness coefficients have a small deviation especially for the term K_{HR}. As shown in Figure 3d, for free-head pile, we observe that the three-parametric model overestimates the swaying coefficient comparing to other solutions. Regarding the stiffness ratio, it is obvious that increasing the ratio E_p/E_s leads to the increase of stiffness.

In Figure 4, a comparison of maximum bending moments at the pile head and at depth $z=\pi/2\lambda$ is presented, in comparison to those presented in literature (Randolph 1981). There is good agreement of the solution of Randolph with the equivalent analytical solution of the three-parametric Winkler model for small soil-pile stiffness ratios (from 10^2-10^3) compared to the classic Winkler model. For greater values of this ratio, the numerical solution approaches the one of the one-parameter Winkler model.

7. CONCLUSIONS

The main conclusions of this paper are summarized below:

- The proposed model contains three soil constants and consequently can simultaneously reproduce all of the three stiffness coefficients at the pile head, in contrast to the one-parameter model which is usually calibrated, so that it can reproduce only horizontal stiffness.
- The proposed method improves the estimation of maximum bending moments for laterally loaded piles in comparison to the classic one-parameter model.
- All the above are achieved with no significant increase in the complexity of the analysis, since the order of the differential equation and the boundary conditions at the pile head and tip do not change compared to the typical Winkler model.

8. ACKNOLEGMENTS

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